

## SDEs driven by a time-changed Lévy process and their associated time-fractional order pseudo-differential equations

Marjorie Hahn · Kei Kobayashi ·  
Sabir Umarov

Submitted: January 17, 2010; Revised: April 6, 2010

**Abstract** It is known that the transition probabilities of a solution to a classical Itô stochastic differential equation (SDE) satisfy in the weak sense the associated Kolmogorov equation. The Kolmogorov equation is a partial differential equation with coefficients determined by the corresponding SDE. Time-fractional Kolmogorov type equations are used to model complex processes in many fields. However, the class of SDEs that is associated with these equations is unknown except in a few special cases. The present paper shows that in the cases of either time-fractional order or more general time-distributed order differential equations, the associated class of SDEs can be described within the framework of SDEs driven by semimartingales. These semimartingales are time-changed Lévy processes where the independent time-change is given respectively by the inverse of a single or mixture of independent stable subordinators. Examples are provided, including a fractional analogue of the Feynman-Kac formula.

**Keywords** time-change · stochastic differential equation · semimartingale · Kolmogorov equation · fractional order differential equation · pseudo-differential operator · Lévy process · stable subordinator

**Mathematics Subject Classification (2000)** 60H10 · 35S10 · 60G51

---

M. Hahn

Department of Mathematics, Tufts University, 503 Boston Avenue, Medford, MA 02155

Tel.: 1-617-627-2363 Fax: 1-617-627-3966

E-mail: marjorie.hahn@tufts.edu

K. Kobayashi

Department of Mathematics, Tufts University, Medford, MA 02155

E-mail: kei.kobayashi@tufts.edu

S. Umarov

Department of Mathematics, Tufts University, Medford, MA 02155

E-mail: sabir.umarov@tufts.edu

## 1 Introduction

In the last few decades, fractional Kolmogorov or equivalently fractional Fokker-Planck equations have appeared as an essential tool for the study of dynamics of various complex stochastic processes arising in anomalous diffusion in physics [19,31], finance [8], hydrology [2], and cell biology [22]. Complexity includes phenomena such as the presence of weak or strong correlations, different sub- or super-diffusive modes, and jump effects. For example, experimental studies of the motion of macromolecules in a cell membrane show apparent subdiffusive motion with several simultaneous diffusive modes (see [22]).

The present paper identifies a wide class of stochastic differential equations (SDEs) whose associated partial differential equations are represented by time-fractional order pseudo-differential equations. This connection provides stochastic processes whose dynamics correspond to time-fractional order pseudo-differential equations.

Let  $B_t$  be an  $m$ -dimensional Brownian motion defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with a complete right-continuous filtration  $(\mathcal{F}_t)$ . A deep connection between a stochastic process and its associated partial differential equation is expressed through the Kolmogorov forward and backward equations [1]. This concept is based on the relationship between two main components: (i) the Cauchy problem

$$\frac{\partial u(t, x)}{\partial t} = \mathcal{A}u(t, x), \quad u(0, x) = \varphi(x), \quad t > 0, x \in \mathbb{R}^n, \quad (1.1)$$

where  $\mathcal{A}$  is the differential operator

$$\mathcal{A} = \sum_{j=1}^n b_j(x) \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{i,j=1}^n \sigma_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \quad (1.2)$$

with coefficients  $b_j(x)$  and  $\sigma_{i,j}(x)$  satisfying some regularity conditions; and (ii) the associated class of Itô SDEs given by

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x. \quad (1.3)$$

The coefficients of SDE (1.3) are connected with the coefficients of the operator  $\mathcal{A}$  as follows:  $b(x) = (b_1(x), \dots, b_n(x))$  and  $\sigma_{i,j}(x)$  is the  $(i, j)$ -th entry of the product of the  $n \times m$  matrix  $\sigma(x)$  with its transpose  $\sigma^T(x)$ .

One mechanism for establishing this relationship is via semigroup theory, in which the operator  $\mathcal{A}$  is recognized as the infinitesimal generator of the semigroup  $T_t(\cdot)(x) := \mathbb{E}[(\cdot)(X_t) | X_0 = x]$  (defined, for instance, on the Banach space  $C_0(\mathbb{R}^n)$  with supnorm), i.e.  $\mathcal{A}\varphi(x) = \lim_{t \rightarrow 0} (T_t - I)\varphi(x)/t$ ,  $\varphi \in \text{Dom}(\mathcal{A})$ , the domain of  $\mathcal{A}$ . A unique solution to (1.1) is represented by  $u(t, x) = (T_t\varphi)(x)$ .

Enlarging the class of SDEs in (1.3) to those driven by a Lévy process leads to a generalization of connection (i)–(ii) where the analogous operator on the right-hand side of (1.1) has additional terms corresponding to jump components of the driving process (see [1,24] and references therein). In this case, the operator  $\mathcal{A}$  in (1.2) takes the form  $\mathcal{L}(x, \mathbf{D}_x)$  in (2.14).

A fractional generalization of the Cauchy problem (1.1) with  $\mathcal{A} = \mathcal{L}(x, \mathbf{D}_x)$ , in the sense that the first order time derivative on the left side of equation (1.1) is replaced by a time-fractional order derivative, has appeared in the framework of continuous time random walks (CTRWs) and fractional kinetic theory [5, 9, 17, 19, 20, 31]. Papers [7, 18, 27, 29] establish that time-fractional versions of the Cauchy problem are connected with limit processes arising from certain weakly convergent sequences or triangular arrays of CTRWs. These limit processes are time-changed Lévy processes, where the time-change arises as the first hitting time of level  $t$  (equivalently, the inverse) for a single stable subordinator.

This paper generalizes the connection (i)–(ii) to time-fractional pseudo-differential equations that imply a fractional analogue of Kolmogorov equations. First, it establishes the class of SDEs replacing (1.3) which is associated with the following Cauchy problem:

$$\mathbf{D}_*^\beta u(t, x) = \mathcal{L}(x, \mathbf{D}_x)u(t, x), \quad u(0, x) = \varphi(x), \quad t > 0, \quad x \in \mathbb{R}^n, \quad (1.1')$$

where  $\mathbf{D}_*^\beta$  is the fractional derivative in the sense of Caputo with  $\beta \in (0, 1)$  (see Section 2), and  $\mathcal{L}(x, \mathbf{D}_x)$  is the pseudo-differential operator in (2.14). The driving processes of the associated class of SDEs are Lévy processes composed with the inverse of a  $\beta$ -stable subordinator,  $\beta \in (0, 1)$  (Theorem 3.5 for  $N = 1$ ). Since such processes are semimartingales, SDEs with respect to them are meaningful and have the form in (3.20). A partial result when the driving process is either Brownian or Lévy stable motion with drift time-changed by the inverse of a single stable subordinator is considered in [14, 15] without specifying the explicit form of the corresponding SDEs.

More generally, the class of SDEs in the above discussion when the time-change process is the inverse of an arbitrary mixture of independent stable subordinators gives rise to a Cauchy problem with a fractional derivative with distributed orders (see (2.3)) on the left of (1.1'), namely,

$$\mathcal{D}_\mu u(t, x) = \mathcal{L}(x, \mathbf{D}_x)u(t, x).$$

In this case, the time-change process is no longer the inverse of a stable subordinator if at least two different indices arise in the mixture. Moreover, SDEs corresponding to time-fractional Kolmogorov equations cannot be described within the classical Brownian- or Lévy-driven SDEs.

Section 2 of this paper recalls the required auxiliary facts. Section 3 formulates and proves the main results of the paper, and provides examples, including a fractional analogue of the Feynman-Kac formula. Section 4 illustrates an alternative technique for establishing the main results in the case of Brownian motion.

## 2 Preliminaries and auxiliaries

The *fractional integral of order*  $\beta > 0$  is

$$J^\beta g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-u)^{\beta-1} g(u) du, \quad t > 0, \quad (2.1)$$

where  $\Gamma(\cdot)$  is Euler's gamma function. By convention,  $J^0 = I$ , the identity operator, and  $Jg(t) := J^1g(t)$ , the integration operator. The *fractional derivative of order  $\beta \in (0, 1)$  in the sense of Caputo* is  $\mathbf{D}_*^\beta g(t) = J^{1-\beta} \frac{d}{dt} g(t)$ ,  $t > 0$ . By convention, set  $\mathbf{D}_*^\beta = \frac{d}{dt}$  for  $\beta = 1$ . The Laplace transform of  $\mathbf{D}_*^\beta g$  is ([6])

$$\widehat{[\mathbf{D}_*^\beta g]}(s) = s^\beta \tilde{g}(s) - s^{\beta-1} g(0+), \quad (2.2)$$

where  $\tilde{g}(s) \equiv \mathcal{L}[g](s) = \int_0^\infty g(t)e^{-st}dt$ , the Laplace transform of  $g$ .

Let  $\mu$  be a finite measure on  $[0, 1]$ . The *fractional derivative with distributed orders* is the operator (see, e.g., [28])

$$\mathcal{D}_\mu g(t) = \int_0^1 \mathbf{D}_*^\beta g(t) d\mu(\beta). \quad (2.3)$$

These operators provide a generalization of fractional order derivatives. The mapping  $\beta \mapsto \mathbf{D}_*^\beta g(t)$  is continuous on  $[0, 1)$  for a differentiable function  $g$ . For example, if  $\mu = C_1\delta_{\beta_1} + C_2\delta_{\beta_2}$ , then  $\mathcal{D}_\mu g(t) = C_1\mathbf{D}_*^{\beta_1} g(t) + C_2\mathbf{D}_*^{\beta_2} g(t)$ .

If a function  $\psi(x, \xi) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}$  is continuous and satisfies a suitable growth condition as  $|\xi| \rightarrow \infty$ , then for  $u \in C_0^\infty(\mathbb{R}^n)$ , the operator

$$\mathcal{A}u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \psi(x, \xi) \hat{u}(\xi) e^{i(x, \xi)} d\xi, \quad x \in \mathbb{R}^n, \quad (2.4)$$

where  $\hat{u}(\xi) = \int_{\mathbb{R}^n} u(x) e^{-i(x, \xi)} dx$ , is meaningful and called a *pseudo-differential operator* with *symbol*  $\psi(x, \xi)$ . For properties of pseudo-differential operators, see the monographs [10, 11, 25].

Pseudo-differential operators of interest in this paper are infinitesimal generators of strongly continuous semigroups constructed from stochastic processes which are solutions to SDEs driven by a Lévy process. Such processes are Feller processes, and therefore, they have strongly continuous semigroups (see [1]). To this end, we will consider symbols  $\psi(x, \xi)$  which are continuous in  $x \in \mathbb{R}^n$  and  $\xi \in \mathbb{R}^n$  and, for each fixed  $x$ , both hermitian and conditionally positive definite in  $\xi$  (see [1] for details).

A *Lévy process*  $L_t \in \mathbb{R}^n$ ,  $t \geq 0$ , with  $L_0 = 0$ , is an adapted càdlàg process with independent stationary increments such that for all  $\epsilon, t > 0$ ,  $\lim_{s \rightarrow t} \mathbb{P}(|L_t - L_s| > \epsilon) = 0$ . Lévy processes are characterized by three parameters: a vector  $b \in \mathbb{R}^n$ , a nonnegative definite matrix  $\Sigma$ , and a measure  $\nu$  defined on  $\mathbb{R}^n \setminus \{0\}$  such that  $\int \min(1, |x|^2) d\nu < \infty$ , called its Lévy measure. The Lévy-Khintchine formula characterizes a Lévy process (as an infinitely divisible process) in terms of its characteristic function  $\Phi_t(\xi) = e^{t\Psi(\xi)}$ , with

$$\Psi(\xi) = i(b, \xi) - \frac{1}{2}(\Sigma\xi, \xi) + \int_{\mathbb{R}^n \setminus \{0\}} (e^{i(w, \xi)} - 1 - i(w, \xi)\chi_{(|w| \leq 1)}(w)) \nu(dw). \quad (2.5)$$

The function  $\Psi$  is called the *Lévy symbol* of  $L_t$  (see, e.g. [1, 21].)

Particularly important for this paper is the class of stable subordinators. For  $\beta \in (0, 1)$ , a  $\beta$ -*stable subordinator* is a one-dimensional strictly increasing Lévy process  $D_t$  starting at 0 which is self-similar, i.e.  $\{D_{at}, t \geq 0\}$  has

the same finite-dimensional distributions as  $\{a^{1/\beta}D_t, t \geq 0\}$ , and the Laplace transform for  $D_1$  is given by

$$\mathbb{E}[e^{-sD_1}] = e^{-s^\beta}, \quad s \geq 0. \quad (2.6)$$

It follows from the general theory of Laplace transforms (see, e.g. [30]) that the density  $f_{D_1}(\tau)$  of  $D_1$  is infinitely differentiable on  $(0, \infty)$ , with the following asymptotics at zero and infinity [16, 26]:

$$f_{D_1}(\tau) \sim \frac{\left(\frac{\beta}{\tau}\right)^{\frac{2-\beta}{2(1-\beta)}}}{\sqrt{2\pi\beta(1-\beta)}} e^{-(1-\beta)\left(\frac{\tau}{\beta}\right)^{-\frac{\beta}{1-\beta}}}, \quad \tau \rightarrow 0; \quad (2.7)$$

$$f_{D_1}(\tau) \sim \frac{\beta}{\Gamma(1-\beta)\tau^{1+\beta}}, \quad \tau \rightarrow \infty. \quad (2.8)$$

Consider an SDE driven by a Lévy process

$$\begin{aligned} Y_t = x + \int_0^t b(Y_{s-})ds + \int_0^t \sigma(Y_{s-})dB_s \\ + \int_0^t \int_{|w|<1} H(Y_{s-}, w) \tilde{N}(ds, dw) + \int_0^t \int_{|w|\geq 1} K(Y_{s-}, w) N(ds, dw), \end{aligned} \quad (2.9)$$

where  $x \in \mathbb{R}^n$ , and the continuous mappings  $b : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ ,  $H : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ , and  $K : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfy the following Lipschitz and growth conditions: there exist positive constants  $C_1$  and  $C_2$  satisfying

$$\begin{aligned} \bullet \quad & |b(y_1) - b(y_2)|^2 + \|\sigma(y_1) - \sigma(y_2)\|^2 + \int_{|w|<1} |H(y_1, w) - H(y_2, w)|^2 \nu(dw) \\ & \leq C_1 |y_1 - y_2|^2, \quad \forall y_1, y_2 \in \mathbb{R}^n; \end{aligned} \quad (2.10)$$

$$\bullet \quad \int_{|w|<1} |H(y, w)|^2 \nu(dw) \leq C_2(1 + |y|^2), \quad \forall y \in \mathbb{R}^n. \quad (2.11)$$

Under these conditions, SDE (2.9) has a unique strong solution  $Y_t$  (see, [1, 24]). If the coefficients  $b$ ,  $\sigma$ ,  $H$ , and  $K$  are bounded, then  $(T_t \varphi)(x) = \mathbb{E}[\varphi(Y_t) | Y_0 = x]$  is a strongly continuous contraction semigroup defined on the Banach space  $C_0(\mathbb{R}^n)$ . Moreover, the pseudo-differential equation associated with the process  $Y_t$  takes the form

$$\frac{\partial u(t, x)}{\partial t} = \mathcal{L}(x, \mathbf{D}_x)u(t, x), \quad (2.12)$$

where the infinitesimal generator  $\mathcal{L}(x, \mathbf{D}_x)$  is a pseudo-differential operator with the symbol

$$\begin{aligned} \Psi(x, \xi) = i(b(x), \xi) - \frac{1}{2}(\Sigma(x)\xi, \xi) \\ + \int_{\mathbb{R}^n \setminus \{0\}} (e^{i(G(x, w), \xi)} - 1 - i(G(x, w), \xi)\chi_{(|w|<1)}(w)) \nu(dw), \end{aligned} \quad (2.13)$$

where  $G(x, w) = H(x, w)$  if  $|w| < 1$ , and  $G(x, w) = K(x, w)$  if  $|w| \geq 1$  ([1, 24]). For each fixed  $x \in \mathbb{R}^n$ , the symbol  $\Psi(x, \xi)$  is continuous, hermitian, and conditionally positive definite [3, 11]. Using  $\mathbf{D}_x = \frac{1}{i}(\partial/\partial x_1, \dots, \partial/\partial x_n)$ , the pseudo-differential operator  $\mathcal{L}(x, \mathbf{D}_x)$  has the form

$$\begin{aligned} \mathcal{L}(x, \mathbf{D}_x)\varphi(x) &= i(b(x), \mathbf{D}_x)\varphi(x) - \frac{1}{2}(\Sigma(x)\mathbf{D}_x, \mathbf{D}_x)\varphi(x) \\ &+ \int_{\mathbb{R}^n \setminus \{0\}} [\varphi(x + G(x, w)) - \varphi(x) - i\chi_{(|w|<1)}(w)(G(x, w), \mathbf{D}_x)\varphi(x)] \nu(dw). \end{aligned} \quad (2.14)$$

Here,  $\mathcal{L}(x, \mathbf{D}_x) : C_0^2(\mathbb{R}^n) \longrightarrow C_0(\mathbb{R}^n)$ , i.e.  $C_0^2(\mathbb{R}^n) \subset \text{Dom}(\mathcal{L}(x, \mathbf{D}_x))$ .

### 3 Main results and examples

Let  $D_t$  be an  $(\mathcal{F}_t)$ -adapted strictly increasing càdlàg process starting at 0 such that  $\lim_{t \rightarrow \infty} D_t = \infty$  a.s. The *inverse* or the *first hitting time process*  $E_t$  of  $D_t$  is defined by  $E_t := \inf\{\tau \geq 0 : D_\tau > t\}$ . The inverse  $E_t$  is a *continuous*  $(\mathcal{F}_t)$ -time-change, i.e. it is a continuous, nondecreasing family of  $(\mathcal{F}_t)$ -stopping times. In fact, for any  $t, \tau > 0$ , we have  $\{E_t < \tau\} = \{D_{\tau-} > t\} \in \mathcal{F}_\tau$ . Hence, by the right-continuity of the filtration  $(\mathcal{F}_t)$ , each random variable  $E_t$  is an  $(\mathcal{F}_t)$ -stopping time.

Let  $D_{1,t}$  and  $D_{2,t}$  be independent  $(\mathcal{F}_t)$ -adapted strictly increasing càdlàg processes. Then  $D_t = D_{1,t} + D_{2,t}$  also possesses the same property, and its inverse process  $E_t$  satisfies  $\mathbb{P}(E_t \leq \tau) = \mathbb{P}(D_\tau > t) = 1 - (F_\tau^{(1)} * F_\tau^{(2)})(t)$ . Here, for  $k = 1, 2$ ,  $F_\tau^{(k)}(t) = \mathbb{P}(D_{k,\tau} \leq t)$  with density  $f_\tau^{(k)}$  (if it exists), and  $*$  denotes convolution of cumulative distribution functions or densities, whichever is required. For notational convenience, if  $a, b > 0$ , let

$$\left[ F_1^{(1)} \left( \frac{\cdot}{a} \right) * F_1^{(2)} \left( \frac{\cdot}{b} \right) \right] (t) := \int_{s=0}^{s=t} F_1^{(1)} \left( \frac{t-s}{a} \right) dF_1^{(2)} \left( \frac{s}{b} \right),$$

which, if the density functions exist, can also be written as

$$\left[ F_1^{(1)} \left( \frac{\cdot}{a} \right) * F_1^{(2)} \left( \frac{\cdot}{b} \right) \right] (t) = \frac{1}{b} \int_{s=0}^{s=t} (Jf_1^{(1)}) \left( \frac{t-s}{a} \right) f_1^{(2)} \left( \frac{s}{b} \right) ds,$$

where  $J$  is the usual integration operator.

**Lemma 3.1** *Let  $D_t = c_1 D_{1,t} + c_2 D_{2,t}$ , where  $c_1, c_2$  are positive constants and  $D_{1,t}$  and  $D_{2,t}$  are independent stable subordinators with respective indices  $\beta_1$  and  $\beta_2$  in  $(0, 1)$ . Then the inverse  $E_t$  of  $D_t$  satisfies*

$$\mathbb{P}(E_t \leq \tau) = 1 - \left[ F_1^{(1)} \left( \frac{\cdot}{c_1 \tau^{1/\beta_1}} \right) * F_1^{(2)} \left( \frac{\cdot}{c_2 \tau^{1/\beta_2}} \right) \right] (t) \quad (3.1)$$

and has the density

$$f_{E_t}(\tau) = -\frac{\partial}{\partial \tau} \left\{ \frac{1}{c_2 \tau^{1/\beta_2}} \left[ (Jf_1^{(1)}) \left( \frac{\cdot}{c_1 \tau^{1/\beta_1}} \right) * f_1^{(2)} \left( \frac{\cdot}{c_2 \tau^{1/\beta_2}} \right) \right] (t) \right\}. \quad (3.2)$$

*Proof* Since  $D_{1,\tau}$  and  $D_{2,\tau}$  are independent and self-similar processes,

$$\mathbb{P}(E_t \leq \tau) = \mathbb{P}(D_\tau > t) = 1 - \mathbb{P}(c_1 \tau^{1/\beta_1} D_{1,1} + c_2 \tau^{1/\beta_2} D_{2,1} \leq t),$$

by which (3.1) follows. Differentiating (3.1) with respect to  $\tau$  yields (3.2).  $\square$

**Lemma 3.2** *For any  $t < \infty$ , the density  $f_{E_t}(\tau)$  in (3.2) is bounded, and there exist a number  $\beta \in (0, 1)$  and positive constants  $C, k$ , not depending on  $\tau$ , such that*

$$f_{E_t}(\tau) \leq C \exp\left(-k\tau^{\frac{1}{1-\beta}}\right) \quad (3.3)$$

for  $\tau$  large enough.

Routine elementary calculations using (2.7) and (2.8) yield Lemma 3.2. This lemma can be extended for processes  $E_t$  which are inverses of stochastic processes of the form  $D_t = \sum_{k=1}^N c_k D_{k,t}$ , where  $D_{k,t}, k = 1, \dots, N$ , are independent stable subordinators of respective indices  $\beta_k \in (0, 1)$ , and  $c_k$  are positive constants.

Theorems 3.1 and 3.2 require the following *assumption*:  $\{T_t, t \geq 0\}$  is a strongly continuous semigroup defined on a Banach space  $\mathcal{X}$  with norm  $\|\cdot\|$ , such that the estimate

$$\|T_t \varphi\| \leq M \|\varphi\| e^{\omega t} \quad (3.4)$$

is valid for some constants  $M > 0$  and  $\omega \geq 0$ . For example, if  $\{T_t\}$  is a semigroup associated with a Feller process, then (3.4) is satisfied with  $\omega = 0$ . This assumption implies that any number  $s$  with  $\operatorname{Re}(s) > \omega$  belongs to the resolvent set  $\rho(\mathcal{A})$  of the infinitesimal generator  $\mathcal{A}$  of  $T_t$  and the resolvent operator is represented in the form  $R(s, \mathcal{A}) = \int_0^\infty e^{-st} T_t dt$  (see [4]).

**Theorem 3.1** *Define the process  $D_t = \sum_{k=1}^N c_k D_{k,t}$ , where  $D_{k,t}, 1 \leq k \leq N$ , are independent stable subordinators with respective indices  $\beta_k \in (0, 1)$  and constants  $c_k > 0$ . Let  $E_t$  be the inverse process to  $D_t$ . Suppose that  $T_t$  is a strongly continuous semigroup in a Banach space  $\mathcal{X}$ , satisfies (3.4), and has infinitesimal generator  $\mathcal{A}$ . Then, for each fixed  $t \geq 0$ , the integral  $\int_0^\infty f_{E_t}(\tau) T_\tau \varphi d\tau$  exists and the vector-function  $v(t) = \int_0^\infty f_{E_t}(\tau) T_\tau \varphi d\tau$ , where  $\varphi \in \operatorname{Dom}(\mathcal{A})$ , satisfies the abstract Cauchy problem*

$$\sum_{k=1}^N C_k \mathbf{D}_*^{\beta_k} v(t) = \mathcal{A}v(t), \quad t > 0, \quad v(0) = \varphi, \quad (3.5)$$

where  $C_k = c_k^{\beta_k}, k = 1, \dots, N$ .

*Proof* For simplicity, the proof will be given in the case  $N = 2$ . First, define the vector-function  $p(\tau) = T_\tau \varphi$ , where  $\varphi \in \operatorname{Dom}(\mathcal{A})$ . In accordance with the conditions of the theorem,  $p(\tau)$  satisfies the abstract Cauchy problem

$$\frac{\partial p(\tau)}{\partial \tau} = \mathcal{A}p(\tau), \quad p(0) = \varphi, \quad (3.6)$$

where  $\mathcal{A}$  is the infinitesimal generator of  $T_\tau$ . Now consider the integral  $\int_0^\infty f_{E_t}(\tau) T_\tau \varphi d\tau$ . It follows from Lemma 3.2 and condition (3.4) that

$$\int_0^\infty f_{E_t}(\tau) \|T_\tau \varphi\| d\tau \leq C \|\varphi\| \int_0^\infty e^{-(k\tau^{\frac{1}{1-\beta}} - \omega\tau)} d\tau < \infty, \quad (3.7)$$

where  $\beta \in (0, 1)$  and  $C, k > 0$  are constants. Hence, the Bochner integral  $\int_0^\infty f_{E_t}(\tau) T_\tau \varphi d\tau$  exists for each fixed  $t \geq 0$ . Denote this vector-function by

$$v(t) := \int_0^\infty f_{E_t}(\tau) T_\tau \varphi d\tau. \quad (3.8)$$

The definition of  $T_t$  implies  $v(0) = \lim_{t \rightarrow 0+} \int_0^\infty f_{E_t}(\tau) T_\tau \varphi d\tau = T_0 \varphi = \varphi$ , in the norm of  $\mathcal{X}$ . By (3.2),

$$v(t) = - \int_0^\infty \frac{\partial}{\partial \tau} \left\{ \frac{1}{c_2 \tau^{1/\beta_2}} \left[ (Jf_1^{(1)}) \left( \frac{\cdot}{c_1 \tau^{1/\beta_1}} \right) * f_1^{(2)} \left( \frac{\cdot}{c_2 \tau^{1/\beta_2}} \right) \right] (t) \right\} T_\tau \varphi d\tau.$$

Since

$$\begin{aligned} \mathcal{L} \left[ \frac{1}{b} (Jf_1^{(1)}) \left( \frac{t}{a} \right) * f_1^{(2)} \left( \frac{t}{b} \right) \right] (s) &= \frac{1}{b} \frac{1}{as} \left( \widetilde{af_1^{(1)}}(as) \right) \left( \widetilde{bf_1^{(2)}}(bs) \right) \\ &= \frac{1}{s} \widetilde{f_1^{(1)}}(as) \widetilde{f_1^{(2)}}(bs), \end{aligned}$$

using (2.6), the Laplace transform of  $v(t)$  takes the form

$$\begin{aligned} \tilde{v}(s) &= - \int_0^\infty \frac{\partial}{\partial \tau} \left\{ \frac{1}{s} e^{-\tau c_1^{\beta_1} s^{\beta_1}} e^{-\tau c_2^{\beta_2} s^{\beta_2}} \right\} T_\tau \varphi d\tau \\ &= (c_1^{\beta_1} s^{\beta_1-1} + c_2^{\beta_2} s^{\beta_2-1}) [\widetilde{T_\tau \varphi}] (c_1^{\beta_1} s^{\beta_1} + c_2^{\beta_2} s^{\beta_2}) \\ &= (C_1 s^{\beta_1-1} + C_2 s^{\beta_2-1}) \tilde{p}(C_1 s^{\beta_1} + C_2 s^{\beta_2}), \end{aligned} \quad (3.9)$$

where  $C_k = c_k^{\beta_k}$ ,  $k = 1, 2$ . Due to (3.4), this is well defined for all  $s$  such that  $C_1 s^{\beta_1} + C_2 s^{\beta_2} > \omega$ . On the other hand it follows from (3.6) that

$$(s - \mathcal{A}) \tilde{p}(s) = \varphi, \quad \forall s > \omega. \quad (3.10)$$

Let  $\omega_0 \geq 0$  be a number such that  $s > \omega_0$  iff  $C_1 s^{\beta_1} + C_2 s^{\beta_2} > \omega$ . Then (3.9) and (3.10) together yield

$$[C_1 s^{\beta_1} + C_2 s^{\beta_2} - \mathcal{A}] \tilde{v}(s) = (C_1 s^{\beta_1-1} + C_2 s^{\beta_2-1}) \varphi, \quad s > \omega_0.$$

Writing this in the form

$$C_1 [s^{\beta_1} \tilde{v}(s) - s^{\beta_1-1} v(0)] + C_2 [s^{\beta_2} \tilde{v}(s) - s^{\beta_2-1} v(0)] = \mathcal{A} \tilde{v}(s), \quad s > \omega_0,$$

recalling (2.2), and applying the inverse Laplace transform to both sides gives

$$C_1 \mathbf{D}_*^{\beta_1} v(t) + C_2 \mathbf{D}_*^{\beta_2} v(t) = \mathcal{A} v(t).$$

Hence  $v(t)$  satisfies the Cauchy problem (3.5). The case  $N > 2$  can be proved using the same method.  $\square$



The next theorem provides an extension with  $D_t$  as the weighted average of an arbitrary number of independent stable subordinators. It is easy to verify that the process  $D_t = \sum_{k=1}^N c_k D_{k,t}$  given in Theorem 3.1 satisfies

$$\ln \mathbb{E}[e^{-sD_t}] \Big|_{t=1} = - \sum_{k=1}^N c_k^{\beta_k} s^{\beta_k}, \quad s \geq 0. \quad (3.11)$$

The function on the right-hand side of (3.11) can be expressed as the integral  $-\int_0^1 s^\beta d\mu(\beta)$ , with  $\mu$  the finite atomic measure  $\mu = \sum_{k=1}^N c_k^{\beta_k} \delta_{\beta_k}$ . The integral  $\int_0^1 s^\beta d\mu(\beta)$  is meaningful for any finite measure  $\mu$  defined on  $[0, 1]$ . Let  $\mathcal{S}$  designate the class of càdlàg  $(\mathcal{F}_t)$ -adapted strictly increasing processes  $V_t$  whose Laplace transform is given by

$$\mathbb{E}[e^{-sV_t}] = \exp\left[-t \int_0^1 s^\beta d\mu(\beta)\right], \quad s \geq 0, \quad (3.12)$$

where  $\mu$  is a finite measure on  $[0, 1]$ . By construction,  $V_0 = 0$  a.s., and  $V_t$  can be considered as a weighted mixture of independent stable subordinators. For the process  $V_t \in \mathcal{S}$  corresponding to a finite measure  $\mu$ , we use the notation  $V_t = D(\mu; t)$  to indicate this correspondence.

**Theorem 3.2** *Assume that  $D(\mu; t) \in \mathcal{S}$  where  $\mu$  is a positive finite measure with  $\text{supp } \mu \subset (0, 1)$ , and let  $E_t$  be the inverse process to  $D(\mu; t)$ . Then the vector-function  $v(t) = \int_0^\infty f_{E_t}(\tau) T_\tau \varphi d\tau$ , where  $T_t$  and  $\varphi$  are as in Theorem 3.1, exists and satisfies the abstract Cauchy problem*

$$\mathcal{D}_\mu v(t) := \int_0^1 \mathbf{D}_*^\beta v(t) d\mu(\beta) = \mathcal{A}v(t), \quad t > 0, \quad v(0) = \varphi. \quad (3.13)$$

*Proof* We briefly sketch the proof since the idea is similar to the proof of Theorem 3.1. Since  $\text{supp } \mu \subset (0, 1)$ , the density  $f_{D(\mu; t)}(\tau)$ ,  $\tau \geq 0$ , exists and has asymptotics (2.7) with some  $\beta = \beta_0 \in (0, 1)$  and (2.8) with some  $\beta = \beta_1 \in (0, 1)$ . This implies the existence of the vector-function  $v(t)$ . Further, one can readily see that  $v(t) = -\int_0^\infty \frac{\partial}{\partial \tau} \{J f_{D(\mu; \tau)}(t)\} (T_\tau \varphi) d\tau$ . Now it follows from the definition of  $D(\mu; t)$  that the Laplace transform of  $v(t)$  satisfies

$$\tilde{v}(s) = \frac{\int_0^1 s^\beta d\mu(\beta)}{s} \int_0^\infty e^{-\tau \int_0^1 s^\beta d\mu(\beta)} (T_\tau \varphi) d\tau = \frac{\eta(s)}{s} \tilde{p}(\eta(s)), \quad s > \bar{\omega}, \quad (3.14)$$

where  $\eta(s) = \int_0^1 s^\beta d\mu(\beta)$ ,  $p$  is a solution to the abstract Cauchy problem (3.6), and  $\bar{\omega} > 0$  is a number such that  $s > \bar{\omega}$  if  $\eta(s) > \omega$  ( $\bar{\omega}$  is uniquely defined since  $\eta(s)$  is a strictly increasing function). Combining (3.14) and (3.10),

$$(\eta(s) - \mathcal{A})\tilde{v}(s) = \varphi \frac{\eta(s)}{s}, \quad s > \bar{\omega}. \quad (3.15)$$

Applying the Laplace transform to (3.13) yields (3.15), as desired.  $\square$

*Remark 3.1*

- a) Equation (3.13) is called a *distributed order differential equation (DODE)*.
- b) If  $\omega = 0$  in (3.4), that is the semigroup  $T_t$  satisfies the inequality  $\|T_t\| \leq M$ , then the condition  $\text{supp } \mu \subset (0, 1)$  in Theorem 3.2 can be replaced by  $\text{supp } \mu \subset [0, 1)$ .

Suppose that  $L_t$  is an  $(\mathcal{F}_t)$ -adapted Lévy process. Let  $E_t$  be a continuous  $(\mathcal{F}_t)$ -time-change. Consider the following SDE driven by the time-changed Lévy process  $L_{E_t}$ :

$$\begin{aligned} X_t = x &+ \int_0^t b(E_s, X_{s-}) dE_s + \int_0^t \sigma(E_s, X_{s-}) dB_{E_s} \\ &+ \int_0^t \int_{|w| < 1} H(E_s, X_{s-}, w) \tilde{N}(dE_s, dw) + \int_0^t \int_{|w| \geq 1} K(E_s, X_{s-}, w) N(dE_s, dw), \end{aligned} \quad (3.16)$$

with  $x \in \mathbb{R}$ , and continuous maps  $b(u, y) : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $\sigma(u, y) : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ , and  $G(u, y, w) = \chi_{(|w| < 1)}(w)H(u, y, w) + \chi_{(|w| \geq 1)}(w)K(u, y, w) : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfying the Lipschitz and growth conditions (2.10), (2.11) with respect to the variable  $y \in \mathbb{R}^n$ , for each fixed  $u \geq 0$ .

SDE (3.16) is obtained from an SDE driven by a Lévy process upon replacing its driving process  $L_t$  by  $L_{E_t}$ . Since  $L_t$  is an  $(\mathcal{F}_t)$ -semimartingale, it follows from Corollary 10.12 of [12] that  $L_{E_t}$  is an  $(\mathcal{F}_{E_t})$ -semimartingale. Thus, (3.16) is the integral form of an SDE driven by an  $(\mathcal{F}_{E_t})$ -semimartingale. We use the following shorthand form for the differential problem given by SDE (3.16):

$$dX_t = F(E_t, X_{t-}) \odot dL_{E_t}, \quad X_0 = x, \quad (3.17)$$

where  $F(u, y) = (b(u, y), \sigma(u, y), G(u, y, \cdot))$  indicates the triple of coefficients controlling the drift, Brownian, and jump terms, respectively. In the future, the integral form in (3.16) will also be expressed using the symbol  $\odot$ .

The SDE in (3.17) is closely related to the following SDE driven by the Lévy process  $L_t$ :

$$dY_\tau = F(\tau, Y_{\tau-}) \odot dL_\tau, \quad Y_0 = x. \quad (3.18)$$

A duality between SDE (3.17) and SDE (3.18) exists as a special case of a general duality result in [13], but what is required here is given below.

**Theorem 3.3** *Let  $D_t$  be a càdlàg  $(\mathcal{F}_t)$ -adapted strictly increasing process, and  $E_t$  be its inverse.*

- 1) *If  $Y_\tau$  satisfies SDE (3.18), then  $X_t := Y_{E_t}$  satisfies SDE (3.17).*
- 2) *If  $X_t$  satisfies SDE (3.17), then  $Y_\tau := X_{D_\tau}$  satisfies SDE (3.18).*

*Proof* Since  $D_t$  is a strictly increasing  $(\mathcal{F}_t)$ -adapted process, its inverse  $E_t$  is a continuous  $(\mathcal{F}_t)$ -time-change. Suppose that  $Y_\tau$  satisfies SDE (3.18) and let  $X_t = Y_{E_t}$ . Then by Proposition 10.21 in [12],

$$X_t = x + \int_0^{E_t} F(s, Y_{s-}) \odot dL_s = x + \int_0^t F(E_s, Y_{E(s)-}) \odot dL_{E_s}. \quad (3.19)$$

$X_t$  will satisfy SDE (3.17), provided that  $X_{s-} = (Y \circ E)_{s-}$  can replace  $Y_{E(s)-}$  in (3.19). The equality  $Y_{E(s)-} = (Y \circ E)_{s-}$  fails only when  $s > 0$  and  $E$  is constant on some closed interval  $[s - \varepsilon, s] \subset (0, t]$  with  $\varepsilon > 0$ . However, the integrator  $L \circ E$  on the right-hand side of (3.19) is constant on this interval. Hence, the difference between the two values  $Y_{E(s)-}$  and  $X_{s-} = (Y \circ E)_{s-}$  does not affect the value of the integral. Consequently, (3.19) is valid with  $X_{s-}$  in place of  $Y_{E(s)-}$ . Thus,  $X_t$  satisfies SDE (3.17), establishing part 1). Similarly, part 2) can be proven using Theorem 3.1 in [13], instead of Proposition 10.21 in [12].  $\square$

**Theorem 3.4** ([1, 24]) *If  $F(u, y)$  satisfies the Lipschitz and growth conditions (2.10) and (2.11) for each  $u \geq 0$ , then SDE (3.18) has a unique strong solution with càdlàg paths.*

**Corollary 3.1** *If  $F(u, y)$  satisfies the Lipschitz and growth conditions (2.10) and (2.11) for each  $u \geq 0$ , then SDE (3.17) has a unique strong solution with càdlàg paths.*

*Proof* A strong solution to SDE (3.17) clearly exists by Theorems 3.3 and 3.4. To prove the uniqueness, notice that the driving process  $L_{E_t}$  of SDE (3.17) is constant on any interval  $[D_{s-}, D_s]$  and so is the solution  $X_t$ . This implies a unique representation  $X_t = X_{D_{E(t)}} = Y_{E_t}$ , where  $Y_\tau$  is a unique strong solution to SDE (3.18).  $\square$

**Theorem 3.5** *Let  $D_{k,t}$ ,  $k = 1, \dots, N$ , be independent stable subordinators of respective indices  $\beta_k \in (0, 1)$ . Define  $D_t = \sum_{k=1}^N c_k D_{k,t}$ , with positive constants  $c_k$ , and let  $E_t$  be its inverse. Suppose that a stochastic process  $Y_\tau$  satisfies the SDE (2.9) driven by a Lévy process, where continuous mappings  $b, \sigma, H, K$  are bounded and satisfy condition (2.10). Let  $X_t = Y_{E_t}$ . Then*

1)  $X_t$  satisfies the SDE driven by the time-changed Lévy process

$$\begin{aligned} X_t = x &+ \int_0^t b(X_{s-}) dE_s + \int_0^t \sigma(X_{s-}) dB_{E_s} \\ &+ \int_0^t \int_{|w| < 1} H(X_{s-}, w) \tilde{N}(dE_s, dw) + \int_0^t \int_{|w| \geq 1} K(X_{s-}, w) N(dE_s, dw); \end{aligned} \quad (3.20)$$

2) if  $Y_\tau$  is independent of  $E_t$ , then the function  $u(t, x) = \mathbb{E}[\varphi(X_t) | X_0 = x]$  satisfies the following Cauchy problem

$$\sum_{k=1}^N C_k \mathbf{D}_*^{\beta_k} u(t, x) = \mathcal{L}(x, \mathbf{D}_x) u(t, x), \quad u(0, x) = \varphi(x), \quad t > 0, x \in \mathbb{R}^n, \quad (3.21)$$

where  $\varphi \in C_0^2(\mathbb{R}^n)$ ,  $C_k = c_k^{\beta_k}$ ,  $k = 1, \dots, N$ , and the pseudo-differential operator  $\mathcal{L}(x, \mathbf{D}_x)$  is as in (2.14) with symbol in (2.13).

*Proof* Again, for simplicity, we give the proof in the case  $N = 2$ .

1) Since  $D_t$  is a linear combination of stable subordinators, which are càdlàg and strictly increasing, it follows that  $D_t$  is also càdlàg and strictly increasing. Hence, it follows from Theorem 3.3 that  $X_t = Y_{E_t}$  satisfies SDE (3.20).

2) Consider  $T_\tau^Y \varphi(x) = \mathbb{E}[\varphi(Y_\tau) | Y_0 = x]$ , where  $Y_\tau$  is a solution of SDE (2.9). Then  $T_\tau^Y$  is a strongly continuous contraction semigroup in the Banach space  $C_0(\mathbb{R}^n)$  (see [1]) which satisfies (3.4) with  $\omega = 0$ , has infinitesimal generator given by the pseudo-differential operator  $\mathcal{L}(x, \mathbf{D}_x)$  with symbol  $\Psi(x, \xi)$  defined in (2.13), and  $C_0^2(\mathbb{R}^n) \subset \text{Dom}(\mathcal{L}(x, \mathbf{D}_x))$ . So the function  $p^Y(\tau, x) = T_\tau^Y \varphi(x)$  with  $\varphi \in C_0^2(\mathbb{R}^n)$  satisfies the Cauchy problem

$$\frac{\partial p^Y(\tau, x)}{\partial \tau} = \mathcal{L}(x, \mathbf{D}_x) p^Y(\tau, x), \quad p^Y(0, x) = \varphi(x). \quad (3.22)$$

Furthermore, consider  $p^X(t, x) = \mathbb{E}[\varphi(X_t) | X_0 = x] = \mathbb{E}[\varphi(Y_{E_t}) | Y_0 = x]$  (recall that  $E_0 = 0$ ). Using the independence of the processes  $Y_\tau$  and  $E_t$ ,

$$p^X(t, x) = \int_0^\infty \mathbb{E}[\varphi(Y_\tau) | E_t = \tau, Y_0 = x] f_{E_t}(\tau) d\tau = \int_0^\infty f_{E_t}(\tau) T_\tau^Y \varphi(x) d\tau. \quad (3.23)$$

Now, in accordance with Theorem 3.1,  $p^X(t, x)$  satisfies the Cauchy problem (3.21).  $\square$

**Theorem 3.6** *Assume that  $D(\mu; t) \in \mathcal{S}$ , where  $\mu$  is a positive finite measure with  $\text{supp } \mu \subset [0, 1)$ , and let  $E_t$  be its inverse. Suppose that a stochastic process  $Y_\tau$  satisfies SDE (2.9), and let  $X_t = Y_{E_t}$ . Then*

- 1)  $X_t$  satisfies SDE (3.20);
- 2) if  $Y_\tau$  is independent of  $E_t$ , then the function  $u(t, x) = \mathbb{E}[\varphi(X_t) | X_0 = x]$  satisfies the following Cauchy problem

$$\mathcal{D}_\mu u(t, x) = \mathcal{L}(x, \mathbf{D}_x) u(t, x), \quad u(0, x) = \varphi(x), \quad t > 0, x \in \mathbb{R}^n. \quad (3.24)$$

*Proof* The proof of part 1) again follows from Theorem 3.3. Part 2) follows from Theorem 3.2 in a manner similar to the proof of part 2) of Theorem 3.5.  $\square$

**Remark 3.2** Theorems 3.5 and 3.6 reveal the class of SDEs which are associated with the wide class of DODE pseudo-differential equations. Each SDE in this class is driven by a semimartingale which is a time-changed Lévy process, where the time-change is given by the inverse of a mixture of independent stable subordinators. Therefore, these SDEs cannot be represented as classical SDEs driven by a Brownian motion or a Lévy process.

**Corollary 3.2** *Let the coefficients  $b, \sigma, H, K$  of the pseudo-differential operator  $\mathcal{L}(x, \mathbf{D}_x)$  defined in (2.14) with symbol in (2.13) be continuous, bounded, and satisfying condition (2.10). Suppose  $\varphi \in C_0^2(\mathbb{R}^n)$ . Then the Cauchy problem*

$$\mathcal{D}_\mu u(t, x) = \mathcal{L}(x, \mathbf{D}_x) u(t, x), \quad u(0, x) = \varphi(x), \quad t > 0, x \in \mathbb{R}^n,$$

*has a unique solution such that  $u(t, x) \in C_0^2(\mathbb{R}^n)$  for each  $t > 0$ .*

*Proof* The result follows from representation (3.23) in conjunction with estimate (3.7).  $\square$

*Example 1. Time-changed  $\alpha$ -stable Lévy process.*

Let  $L_{\alpha,t}$  be a symmetric  $n$ -dimensional  $\alpha$ -stable Lévy process, which is a pure jump process. If  $p^L(t, x) = E[\varphi(L_{\alpha,t}) | L_{\alpha,\tau} = x]$ , where  $\varphi \in C_0^2(\mathbb{R}^n)$ , then  $p^L(t, x)$  satisfies in the strong sense the Cauchy problem ([23])

$$\frac{\partial p^L(t, x)}{\partial t} = -\kappa_\alpha(-\Delta)^{\alpha/2} p^L(t, x), \quad p^L(0, x) = \varphi(x), \quad t > 0, x \in \mathbb{R}^n, \quad (3.25)$$

where  $\kappa_\alpha$  is a constant depending on  $\alpha$ , and  $(-\Delta)^{\alpha/2}$  is a fractional power of the Laplace operator. The operator on the right-hand side of (3.25) can be represented as a pseudo-differential operator with the symbol  $\psi(\xi) := |\xi|^\alpha$ .

Now suppose that  $Y_t$  solves the SDE

$$dY_t = g(Y_{t-})dL_{\alpha,t}, \quad Y_0 = x, \quad (3.26)$$

where  $g(x)$  is a Lipschitz-continuous function. Notice that (3.26) takes the form given in (3.18) with the pure jump process  $L_{\alpha,t}$  as the driving process and  $F(x) = (0, 0, g(x))$ . In this case, the forward Kolmogorov equation takes the form ([23])

$$\frac{\partial p^Y(t, x)}{\partial t} = -\kappa_\alpha(-\Delta)^{\alpha/2} \{[g(x)]^\alpha p^Y(t, x)\}, \quad t > 0, x \in \mathbb{R}^n. \quad (3.27)$$

Application of Theorem 3.6 implies that  $X_t = Y_{E_t}$  satisfies the SDE

$$dX_t = g(X_{t-})dL_{\alpha,E_t}, \quad X_0 = x, \quad (3.28)$$

where  $E_t$  is the first hitting time of the process  $D(\mu; t)$  described in this theorem. Moreover, if  $E_t$  is independent of  $Y_t$ , then the corresponding forward Kolmogorov equation becomes

$$\mathcal{D}_\mu p^X(t, x) = -\kappa_\alpha(-\Delta)^{\alpha/2} \{[g(x)]^\alpha p^X(t, x)\}, \quad t > 0, x \in \mathbb{R}^n, \quad (3.29)$$

where  $\mathcal{D}_\mu$  is the operator defined in (3.13). When the SDE in (3.28) is driven by a nonsymmetric  $\alpha$ -stable Lévy process, an analogue of (3.29) holds using instead of (3.27) its analogue appearing in [23].  $\square$

*Example 2. Fractional analogue of the Feynman-Kac formula.*

Suppose that  $Y_t$  is a strong solution of SDE (2.9). Let  $\bar{Y} \in \mathbb{R}^n$  be a fixed point and  $q$  be a nonnegative continuous function. Consider the process  $Y_t^q$  defined by  $Y_t^q = Y_t$  if  $0 \leq t < \mathcal{T}_q$ , and  $Y_t^q = \bar{Y}$  if  $t \geq \mathcal{T}_q$ , where  $\mathcal{T}_q$  is an  $(\mathcal{F}_t)$ -stopping time satisfying  $\mathbb{P}(\mathcal{T}_q > t | \mathcal{F}_t) = \exp(-\int_0^t q(Y_s)ds)$ . Then  $Y_t^q$  is a Feller process with associated semigroup (see [1])

$$(T_t^q \varphi)(y) = \mathbb{E} \left[ \exp \left( - \int_0^t q(Y_s)ds \right) \varphi(Y_t) \middle| Y_0 = y \right], \quad (3.30)$$

and infinitesimal generator  $\mathcal{L}_q(x, \mathbf{D}_x) = -q(x) + \mathcal{L}(x, \mathbf{D}_x)$ , where  $\mathcal{L}(x, \mathbf{D}_x)$  is the pseudo-differential operator defined in (2.14). Let  $E_t$  be the inverse to a  $\beta$ -stable subordinator independent of  $Y_t$ . Then it follows from Theorem 3.5 with  $N = 1$  that the transition probabilities of the process  $X_t = Y_{E_t}$  solve the Cauchy problem for the fractional order equation

$$\mathbf{D}_*^\beta u(t, x) = [-q(x) + \mathcal{L}(x, \mathbf{D}_x)]u(t, x), \quad u(0, x) = \varphi(x), \quad t > 0, x \in \mathbb{R}^n.$$

Consequently, (3.30), with  $X_t = Y_{E_t}$  replacing  $Y_t$ , represents a fractional analogue of the Feynman-Kac formula.  $\square$

#### 4 Time-changed Itô formula and its application

This section illustrates a new method of derivation of time-fractional differential equations based on the time-changed Itô formula in [13] *without using the duality principle* (Theorem 3.3). For simplicity, we consider only the one-dimensional case with a Brownian motion as the driving process. Let  $A^*$  be the operator defined as

$$A^*h(y) = -\frac{\partial}{\partial y}\{b(y)h(y)\} + \frac{1}{2}\frac{\partial^2}{\partial y^2}\{\sigma^2(y)h(y)\}. \quad (4.1)$$

**Theorem 4.1** *Let  $B_t$  be a one-dimensional standard  $(\mathcal{F}_t)$ -Brownian motion. Let  $D_t = \sum_{k=1}^N c_k D_{k,t}$ , where  $c_k$  are positive constants and  $D_{k,t}$  are stable subordinators of respective indices  $\beta_k \in (0, 1)$ . Let  $E_t$  be the inverse process to  $D_t$ . Suppose that  $X_t$  is a process defined by the SDE*

$$dX_t = b(X_t)dE_t + \sigma(X_t)dB_{E_t}, \quad X_0 = x, \quad (4.2)$$

*where  $b(y)$  and  $\sigma(y)$  satisfy the Lipschitz condition (2.10). Suppose also that  $X_{D_t}$  is independent of  $E_t$ . Then the transition probability  $p^X(t, y|x) \equiv p^X(t, y)$  satisfies in the weak sense the time-fractional differential equation*

$$\sum_{k=1}^N c_k^{\beta_k} \mathbf{D}_*^{\beta_k} p^X(t, y) = A^* p^X(t, y), \quad (4.3)$$

*with initial condition  $p^X(0, y) = \delta_x(y)$ , the Dirac delta function with mass on  $x$ , where  $A^*$  is the operator in (4.1).*

*Proof* For simplicity, the proof is done for  $N = 2$ . Let  $Y_t = X_{D_t}$ . Then it follows that  $X_t = X_{D_{E(t)}} = Y_{E_t}$ , as in the proof of Corollary 3.1. Hence, by the independence assumption between  $Y_t$  and  $E_t$ ,

$$p^X(t, y) = \int_0^\infty p^Y(u, y) f_{E_t}(u) du \quad (4.4)$$

in the sense of distributions.

Since we are not assuming the duality principle (Theorem 3.3), the fact that  $p^Y$  satisfies the classical Kolmogorov equation  $\frac{\partial}{\partial t} p^Y(t, y) = A^* p^Y(t, y)$  cannot be used here. Instead, we employ the time-changed Itô formula to obtain another representation of  $p^X$  in terms of  $p^Y$  as follows. Let  $f \in C_c^\infty(\mathbb{R})$ . Since  $X$  is constant on every interval  $[D_{s-}, D_s]$ , it follows that  $X_{D(s-)} = X_{D_s} = Y_s$  and the time-changed Itô formula in [13] yields

$$\begin{aligned} f(X_t) - f(x) &= \int_0^{E_t} f'(Y_s) b(Y_s) ds + \int_0^{E_t} f'(Y_s) \sigma(Y_s) dB_s \\ &\quad + \frac{1}{2} \int_0^{E_t} f''(Y_s) \sigma^2(Y_s) ds. \end{aligned} \quad (4.5)$$

Because  $f \in C_c^\infty(\mathbb{R})$ , the process  $M$  defined by  $M_u := \int_0^u f'(Y_s) \sigma(Y_s) dB_s$  is an  $(\mathcal{F}_t)$ -martingale. Taking expectations in (4.5) and conditioning on  $E_t$  which has density  $f_{E_t}$  given in (3.2), we have

$$\begin{aligned} \mathbb{E}[f(X_t) | X_0 = x] - f(x) &= \int_0^\infty \mathbb{E} \left[ M_u + \int_0^u \left\{ f'(Y_s) b(Y_s) + \frac{1}{2} f''(Y_s) \sigma^2(Y_s) \right\} ds \middle| E_t = u, Y_0 = x \right] f_{E_t}(u) du \\ &= \int_0^\infty \int_0^u \mathbb{E} \left[ f'(Y_s) b(Y_s) + \frac{1}{2} f''(Y_s) \sigma^2(Y_s) \middle| Y_0 = x \right] ds f_{E_t}(u) du \end{aligned}$$

by the assumption that  $Y_t = X_{D_t}$  is independent of  $E_t$ . The Fubini theorem is allowed since  $f \in C_c^\infty(\mathbb{R})$  and  $b$  and  $\sigma$  are continuous functions. Using  $p^Y$ , the above can be rewritten as

$$\begin{aligned} \mathbb{E}[f(X_t) | X_0 = x] - f(x) &= \int_0^\infty \int_0^u \int_{-\infty}^\infty \left\{ f'(y) b(y) + \frac{1}{2} f''(y) \sigma^2(y) \right\} p^Y(s, y) dy ds f_{E_t}(u) du \\ &= \int_{-\infty}^\infty f(y) \left\{ \int_0^\infty (JA^* p^Y(u, y)) f_{E_t}(u) du \right\} dy, \end{aligned} \quad (4.6)$$

where  $J$  is the integral operator. On the other hand, reexpressing the left-hand side of (4.6) in terms of  $p^X$  yields

$$\mathbb{E}[f(X_t) | X_0 = x] - f(x) = \int_{-\infty}^\infty f(y) p^X(t, y) dy - f(x). \quad (4.7)$$

Since  $f \in C_c^\infty(\mathbb{R})$  is arbitrary and  $C_c^\infty(\mathbb{R})$  is dense in  $L^2(\mathbb{R})$ , comparison of (4.6) and (4.7) leads to another representation of  $p^X$  with respect to  $p^Y$ :

$$p^X(t, y) - \delta_x(y) = \int_0^\infty (JA^* p^Y(u, y)) f_{E_t}(u) du \quad (4.8)$$

in the sense of distributions with  $p^X(0, y) = \delta_x(y)$ .

Now, we use the two representations (4.4) and (4.8) to derive equation (4.3) with the help of Laplace transforms. The Laplace transform of a function  $v(t)$

of the form in (3.8), with  $f_{E_t}$  in (3.2), is computed as in (3.9). Using this fact and taking the Laplace transform of both sides in (4.4), we obtain

$$\widetilde{p^X}(s, y) = (C_1 s^{\beta_1 - 1} + C_2 s^{\beta_2 - 1}) \widetilde{p^Y}(C_1 s^{\beta_1} + C_2 s^{\beta_2}, y), \quad s > 0,$$

where  $C_k = c_k^{\beta_k}$  ( $k = 1, 2$ ); whereas the Laplace transform of (4.8) is

$$\begin{aligned} \widetilde{p^X}(s, y) - \frac{1}{s} \delta_x(y) &= (C_1 s^{\beta_1 - 1} + C_2 s^{\beta_2 - 1}) \widetilde{JA^* p^Y}(C_1 s^{\beta_1} + C_2 s^{\beta_2}, y) \\ &= \frac{C_1 s^{\beta_1 - 1} + C_2 s^{\beta_2 - 1}}{C_1 s^{\beta_1} + C_2 s^{\beta_2}} \widetilde{A^* p^Y}(C_1 s^{\beta_1} + C_2 s^{\beta_2}, y), \quad s > 0. \end{aligned}$$

Combining these two identities, for  $s > 0$ , we have

$$\begin{aligned} &C_1 (s^{\beta_1} \widetilde{p^X}(s, y) - s^{\beta_1 - 1} \delta_x(y)) + C_2 (s^{\beta_2} \widetilde{p^X}(s, y) - s^{\beta_2 - 1} \delta_x(y)) \\ &= (C_1 s^{\beta_1} + C_2 s^{\beta_2}) \left( \widetilde{p^X}(s, y) - \frac{1}{s} \delta_x(y) \right) \\ &= (C_1 s^{\beta_1 - 1} + C_2 s^{\beta_2 - 1}) \widetilde{A^* p^Y}(C_1 s^{\beta_1} + C_2 s^{\beta_2}, y) = \widetilde{A^* p^X}(s, y), \end{aligned}$$

which coincides with the identity obtained from applying the Laplace transform to both sides of (4.3).  $\square$

*Remark 4.1* If SDE (4.2) contains an additional term  $\rho(X_t)dt$ , then the method used in the proof of Theorem 4.1 does not work since the relationship  $Y_{E_t} = X_t$  does not always follow. Example 5.4 in [13] yields the following conjecture: if an additional term  $\rho(X_t)dt$  is included in SDE (4.2), where  $\rho(y)$  also satisfies the Lipschitz condition, then it is expected that the partial differential equation corresponding to (4.3) may involve a fractional integral term.

**Acknowledgements** The authors are indebted to Rudolf Gorenflo, Meredith Burr, Jamison Wolf, and Xinxin Jiang for references and helpful comments. We also appreciate suggestions of an anonymous referee that resulted in a succinct paper.

## References

1. Applebaum, D. Lévy Processes and Stochastic Calculus. Cambridge University Press (2004).
2. Benson, D. A., Wheatcraft, S. W., and Meerschaert, M. M. Application of a fractional advection-dispersion equation. Water Resour. Res. 36(6) 1403–1412 (2000).
3. Courrège, P. Générateur infinitésimal d'un semi-groupe de convolution sur  $\mathbb{R}^n$ , et formule de Lévy-Khinchine. Bull. Sci. Math. (2) 88 3–30 (1964).
4. Engel, K.-J. and Nagel, R. One-parameter Semigroups for Linear Evolution Equations. Springer (1999).
5. Gillis J. E. and Weiss, G. H. Expected number of distinct sites visited by a random walk with an infinite variance. J. Mathematical Phys. 11 1307–1312 (1970).
6. Gorenflo, R. and Mainardi, F. Fractional calculus: integral and differential equations of fractional order. In A. Carpinteri and F. Mainardi (editors): Fractals and Fractional Calculus in Continuum Mechanics. Springer. 223–276 (1997).
7. Gorenflo, R. and Mainardi, F. Random walk models for space-fractional diffusion processes. Fract. Calc. Appl. Anal. 1, (2), 167–191 (1998).



8. Gorenflo, R., Mainardi, F., Scalas, E. and Raberto, M. Fractional calculus and continuous-time finance. III. Mathematical Finance. 171–180. Trends Math., Birkhäuser, Basel (2001).
9. Gorenflo, R., Mainardi, F. and Vivoli, A. Continuous time random walk and parametric subordination in fractional diffusion. Chaos, Solitons Fractals. 34, (1), 87–103 (2007).
10. Hörmander, L. The Analysis of Linear Partial Differential Operators. II. Differential Operators with Constant Coefficients. Springer-Verlag, Berlin (1983).
11. Jacob, N. Pseudo Differential Operators and Markov Processes. Vol. II. Generators and their Potential Theory. Imperial College Press, London (2002).
12. Jacod, J. Calcul Stochastique et Problèmes de Martingales. Lecture Notes in Mathematics, 714. Springer, Berlin (1979).
13. Kobayashi, K. Stochastic calculus for a time-changed semimartingale and the associated stochastic differential equations. arXiv:0906.5385v1 [math.PR] (2009).
14. Magdziarz, M. and Weron, A. Competition between subdiffusion and Lévy flights: a Monte Carlo approach. Phys. Rev. E 75, 056702 (2007).
15. Magdziarz, M., Weron, A. and Klafter, J. Equivalence of the fractional Fokker-Planck and subordinated Langevin equations: the case of a time-dependent force. Phys. Rev. Lett. 101, 210601 (2008).
16. Mainardi, F., Luchko, Y., and Pagnini, G. The fundamental solution of the space-time fractional diffusion equation. Fract. Calc. Appl. Anal. 4, (2), 153–192 (2001).
17. Meerschaert, M. M. and Scheffler, H-P. Limit Distributions for Sums of Independent Random Vectors. Heavy Tails in Theory and Practice. John Wiley and Sons, Inc. (2001).
18. Meerschaert, M. M. and Scheffler, H-P. Triangular array limits for continuous time random walks. Stochastic Process. Appl. 118, 1606–1633 (2008).
19. Metzler, R. and Klafter, J. The random walk’s guide to anomalous diffusion: a fractional dynamics approach. Phys. Rep. 339, (1), 1–77 (2000).
20. Montroll E. W. and Weiss G. H. Random walks on lattices. II. J. Mathematical Phys. 6, 167–181 (1965).
21. Sato, K-i. Lévy Processes and Infinitely Divisible Distributions. Cambridge University Press (1999).
22. Saxton, M. J. and Jacobson, K. Single-particle tracking: applications to membrane dynamics. Annu. Rev. Biophys. Biomol. Struct. 26, 373–399 (1997).
23. Schertzer, D., Larchevêque, M., Duan, J., Yanovsky, V. V., and Lovejoy, S. Fractional Fokker-Planck equation for nonlinear stochastic differential equations driven by non-Gaussian Lévy stable noises. J. Math. Phys. 42, (1), 200–212 (2001).
24. Situ, R. Theory of Stochastic Differential Equations with Jumps and Applications: Mathematical and Analytical Techniques with Applications to Engineering, Springer (2005).
25. Taylor, M. Pseudodifferential Operators. Princeton University Press (1981).
26. Uchaikin, V. V. and Zolotarev, V. M. Chance and Stability. Stable Distributions and their Applications. VSP, Utrecht (1999).
27. Umarov, S. and Gorenflo, R. On multi-dimensional random walk models approximating symmetric space-fractional diffusion processes. Fract. Calc. Appl. Anal. 8, (1), 73–88 (2005).
28. Umarov, S. and Gorenflo, R. Cauchy and nonlocal multi-point problems for distributed order pseudo-differential equations. I. Z. Anal. Anwendungen 24, (3), 449–466 (2005).
29. Umarov, S. and Steinberg, S. Random walk models associated with distributed fractional order differential equations. IMS Lecture Notes Monogr. Ser. High Dimensional Probability. 51, 117–127 (2006).
30. Widder, D. V. The Laplace transform. Princeton University Press (1941).
31. Zaslavsky, G. M. Chaos, fractional kinetics, and anomalous transport. Phys. Rep. 371, (6), 461–580 (2002).